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Analysis of Economic Models Through Calculus of Variations

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**ANALYSIS OF ECONOMIC MODELS
THROUGH CALCULUS OF VARIATIONS**

A Thesis
Presented to
Faculty of the Department of Mathematics
Western Kentucky University
Bowling Green, Kentucky

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

By
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May 2005

ANALYSIS OF ECONOMIC MODELS
THROUGH CALCULUS OF VARIATION

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My aim was to mesh two fields, and Dr. Alex Lebedinsky has helped me in achieving this goal. He through his patience and cooperation helped me understand all that was required for the economic modeling part. His suggestion has made my thesis work more interesting and useful.

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ABSTRACT

This thesis is a combination of two science fields: Mathematics and Economics. Mathematics is often used to formulate a clear and concise solution to economic problems. In my observation calculus of variation has often been used in various macroeconomic problems. This mathematical method deals with maximizing or minimizing of various objective functions given a set of constraints. This topic brings out one of the best ways to show the relationship between mathematics and economics.

My thesis consists of three parts:

The first chapter contains a review of the calculus of variations. Basic definitions and important conditions have been stated. The aim of this chapter was to set the groundwork for understanding calculus of variations so that it can be used in solving various economics models.

In the second chapter we study an economic model from which calculus of variations has been used to solve it. The macroeconomic model deals with optimizing the social welfare

function. The entire working of the model has been discussed and documented in the thesis report.

The third chapter deals with the analysis of the Lucas model which concentrated on how the accumulation of human capital impacts the growth rate of the economy. Lucas assumes that the growth rate of the human capital is linearly related to its level. If we abandon this assumption, will the optimal value of the time devoted to education in the steady state exist? If it exists, will it be same or different?

So we introduced a new model in which the only modification we made to the Lucas model was in the equation that describes the process of human accumulation by introducing a nonlinear component. On investigation of this new model we have shown that it is possible that optimal behavior for an individual can be not to educate himself.

CHAPTER 1

CALCULUS OF VARIATIONS

1.1 INTRODUCTION

Optimization is a universal human goal. Typically, we try to maximize profit, minimize cost, spend least effort to get the work done, etc. This natural propensity to optimize has led to a long standing effort to systematically determine the optimal realization of a variety of activities in the fields of economics, science and engineering. This continuing effort has created a body of mathematical methods called *optimization*, and one of the mathematical tools used to solve such problems is known as *calculus of variations*.

Thus, we can say that calculus of variations deals with problems of maxima and minima. The basic difference between calculus and calculus of variations is that while in calculus the problem is to determine the values of the independent variables for which a given function of these variables takes a maximum or minimum value, in calculus of variations integrals involving one or more unknown functions are considered, and it is required to determine these unknown functions such that the integral shall take maximum or minimum values. [8, Page 4]

Therefore in variational calculus problems the aim is to find the optimum of functions of functions that are called *functionals*.

1.2 FUNCTIONALS

Definition 1.1[5, Page 318]

Let S be a set of functions and let R be the set of real numbers. A mapping F from S to R is called a *functional* on S .

Example1.1

$$F[f] = \int_0^1 f^2(x) dx$$

If $f(x) = x$ then $F[f] = \frac{1}{3}$.

If $f(x) = 1$ then $F[f] = 1$.

For calculus of variation the *functional* is an *integral*, and the function that appears in the integrand of the integral is to be selected to maximize and minimize the value of the integral.

Example1.2 [1, Page 5]

Suppose A and B are two fixed points with coordinates (a, y_a) and (b, y_b) and consider a set of curves

$$Y = Y(x),$$

joining A and B . We need to find a member $y \in Y$ of this set which maximizes the integral

$$J[y(x)] = \int_a^b F(x, y, y') dx. \quad (1)$$

The functional $J[y(x)]$ can be illustrated through the following examples:

$$1. J[y(x)] = \int_a^b \sqrt{1 + (y')^2} dx$$

$$2. J[y(x)] = \int_a^b 2\pi y \sqrt{1 + (y')^2} dx$$

$$3. J[y(x)] = \int_a^b \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}} dx$$

The problem is to minimize the integral $J[y(x)]$ over the set of functions that pass through the end points. This problem is called the ***fundamental problem*** of calculus of variations.

1.3 CLASSIFICATION OF PROBLEMS

Some of the problems that are studied in calculus of variations have a long history going back to ancient Greek times and form the solid base for the basic questions that we will encounter while studying this topic. The importance of these classic problems is even more relevant today as analogous problems in various fields are being encountered. For example, the economic problem that every nation faces is to increase its national income from Y to Y^* in the shortest time possible. This problem is very similar to the Brachistochrone problem [Section 1.4.3], which was studied in the 14th century.

The four basic problems of calculus of variations are given in each of the following subsections.

1.3.1 The Shortest Path Problem. [1, Page 1]

One of the earliest problems has been to find the shortest distance between two points in a plane. Suppose A and B are two fixed points with coordinates (a, y_a) and (b, y_b) and consider a set of curves

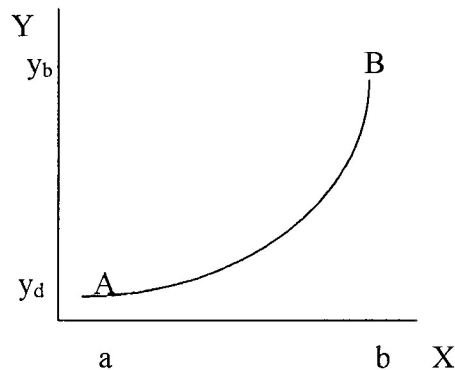
$$Y=Y(x),$$

joining A and B is given by the integral

$$J[y(x)] = \int_a^b \sqrt{1 + (y')^2} dx.$$

1.3.2 The Minimal Surface Problem [1, Page 2]

Given two points, we are to select from the set of functions passing through each point with continuous derivative which produces a surface with the minimum area when rotated about the x-axis.

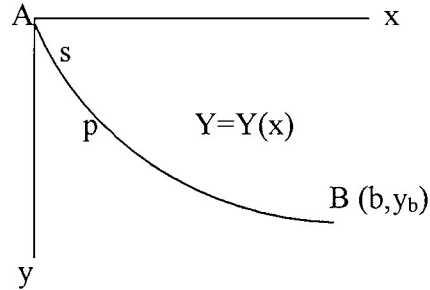


If the curve $y(x)$ in the figure above is rotated about the x-axis, a surface is generated which has surface area

$$J[y(x)] = \int_a^b 2\pi y \sqrt{1 + (y')^2} dx$$

1.3.3 The Brachistochrone Problem [1, Page 3]

This problem involves finding the shape of a smooth wire, joining two points in a vertical plane, down which a small bead will travel in minimum time.



Using the figure above, the speed of the bead P is

$$v = \frac{ds}{dt}.$$

Hence the time traveled from A to B is

$$T(Y) = \int_{x=0}^{x=b} \frac{ds}{v}.$$

Now

$$ds = \sqrt{1 + y'(x)^2} dx,$$

and from dynamics, if a particle starts from rest at A,

$$v^2 = 2gy$$

where g is the acceleration due to gravity. So the integral to be minimized is

$$J[y(x)] = \int_a^b \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}} dx.$$

1.3.4 The Isoperimetric Problem [1, Page 4]

In this problem we must find the maximum of the functional

$$J[y] = \int_0^a y dx$$

subject to the length of the curve being fixed, that is,

$$K[y] = \int_0^a \sqrt{1 + (y')^2} dx.$$

The most famous technique for solving these variational problems is use of the ***Euler-Lagrange equation***.

1.4 THE EULER-LAGRANGE EQUATION [7]

We know that calculus of variations deals with the maximizing and minimizing of functions of functions (called ***functionals***). The strategy used here is similar to that used in calculus where we find the critical points by setting $f'(x) = 0$ and solving for x . In calculus of variation, things are a bit more subtle, since the argument is not a number but a function. However the underlying philosophy is the same as we take the *functional derivative* with respect to the function $y(x)$ and set it equal to zero. This new equation is analogous to the equation

$$\frac{df(x)}{dx} = 0$$

from calculus, but now this differential equation is known as the ***Euler-Lagrange equation***.

Definition: 1.2 [4, Page 372]]

If

$$W = \int_t^T F(x, \frac{dx}{dt}, t) dt \quad \text{and} \quad \frac{d^i x}{dt^i} = 0 \quad \text{for all } i \geq 2$$

then the solution is given by the *Euler- Lagrange equation*:

$$\frac{dF}{dx} - \frac{d}{dt} \frac{dF}{dx'} = 0.$$

Example 1.3: [1, Page 14]

In this example we will solve the *shortest path problem* discussed in Section 1.3.1 earlier.

For $F[y(x)] = \int_{x_0}^{x_1} \sqrt{1 + [y'(x)]^2} dx$, the *Euler Equation* is given by:

$$\begin{aligned} 0 &= \frac{df}{dy} - \frac{d}{dx} \left(\frac{df}{dy'} \right) \\ &= 0 - \frac{d}{dx} \left(\frac{y'(x)}{\sqrt{1 + [y'(x)]^2}} \right) \\ &= - \frac{\sqrt{1 + [y'(x)]^2} y'' - [y'(x)]^2 y''(x) (1 + [y'(x)]^2)^{-\frac{1}{2}}}{1 + [y'(x)]^2} \\ &= - \frac{(1 + [y'(x)]^2) y''(x) - [y'(x)]^2 y''(x)}{(1 + [y'(x)]^2)^{\frac{3}{2}}} \\ &= - \frac{y''(x)}{(1 + [y'(x)]^2)^{\frac{3}{2}}} \end{aligned}$$

Simplifying gives

$$y''(x) = 0.$$

Integrating the above equation twice we obtain

$$y=C_1x+C_2.$$

The above represents the equation of a straight line and thus that will be the shortest path between two points.

Example 1.4[1, Page14]

In this example we will solve the *minimal surface problem*.

For this problem

$$F[y(x)] = \int_a^b 2\pi y \sqrt{1 + (y')^2} dx$$

and the corresponding Euler-Lagrange equation for the critical curve y is

$$\sqrt{1 + y'^2} - \frac{d}{dx} \left\{ \frac{yy'}{\sqrt{1 + y'^2}} \right\} = 0$$

$$(1 + y'^2)^{-\frac{3}{2}} \{1 + y'^2 + yy''\} = 0$$

$$1 + y'^2 = yy''.$$

Using the identity $y'' = y' \frac{dy'}{dy}$, we can write the above equation in separable form

$$\frac{dy}{y} = \frac{1}{2} \frac{d(y'^2)}{(1 + y'^2)}.$$

Integrating we get

$$y=C(\sqrt{1 + y'^2}),$$

where C =Constant.

Solving for y' and integrating again we obtain

$$y = C \cosh\left(\frac{x + C_1}{C}\right).$$

This critical curve is an arc of a *catenary*, where the constants C and C_1 are determined by the **boundary conditions** $y(a) = y_a$ and $y(b) = y_b$.

1.5 BOUNDARY CONDITIONS

Boundary conditions are a set of mathematical conditions to be satisfied. In calculus of variation boundary conditions are the initial and the terminal points through which a function must pass in order to achieve the maxima or minima.

Definition 1.3 [5, Page 331]

The initial and the terminal conditions that serve to constrain the problem are called the boundary conditions.

There are four combinations that can be set as boundary conditions in variational problems.

Fixed Point-Fixed Point

In this case both the initial and the terminal point are fixed. Let the initial point be $P_1 = (a, y(a))$ and the terminal point $P_2 = (b, y(b))$. In order to maximize or minimize the function $y_0(x)$ is required to pass through the points $y_0(a) = y(a)$ and $y_0(b) = y(b)$. The *minimal surface of revolution*, *Brachistochrone* and *isoperimetric* problems are *examples* of problems which have both the initial and terminal points as fixed.

Example 1.5[6, Page 50]

In Example 1.4 we calculated that the solution of the minimal surface of revolution is

$$y = C_0 \cosh \left(\frac{x}{C_0} + C_1 \right).$$

Thus the required minimal surface of revolution (if it exists) is obtained by revolving the appropriate catenary about the x-axis. An appropriate catenary is one which passes through the prescribed end points. This can be done by choosing two constants of integration C_0 and C_1 so that

$$C_0 \cosh \left(\frac{a}{C_0} + C_1 \right) = A$$

$$C_0 \cosh \left(\frac{b}{C_0} + C_1 \right) = B$$

for positive A and B. The solutions of the system of the two transcendental equations for C_0 and C_1 will depend on the magnitude of A and B. We will limit our discussion to the case $a = 0$ and $A = B$. Thus the from the two end conditions

$$\cosh(C_1) = \cosh\left(C_1 + \frac{b}{C_0}\right).$$

This relationship requires

$$C_1 + \frac{b}{C_0} = -C_1 \text{ or}$$

$$C_1 = -\frac{b}{2C_0}.$$

The above equations will yield $y(x) = C_0 \cosh\left(\frac{2x - b}{2C_0}\right)$. Let $\gamma = \frac{b}{2C_0}$. Then,

$$\frac{y}{A} = \frac{1}{\sigma\gamma} \cosh\left(\gamma\left[\frac{2x}{b} - 1\right]\right)$$

where

$$\sigma = \frac{2A}{b}.$$

Then $y(b) = A$ becomes

$$\cosh(\gamma) = \sigma\gamma. \quad (1)$$

For a given A and b , $\sigma = \frac{2A}{b}$ is a known value. Hence the only unknown $\gamma = \frac{b}{2C_0}$ is the

required root for equation (1). Note that

$$\begin{aligned} y(0) &= A \cosh\left(\frac{-\gamma}{\sigma\gamma}\right) \\ &= A \cosh\left(\frac{\gamma}{\sigma\gamma}\right) \\ &= A. \end{aligned}$$

Thus the second boundary condition is also satisfied.

Variable Point-Variable Point [5, Page 331]

Case 1:

Suppose both the end points fall anywhere on the parallel lines $x=a$ and $x=b$. In this case only the x -coordinates of the end points are fixed while the y coordinates are free to vary.

Such problems satisfy the *natural boundary* conditions.

Definition1.4 [5, Page 332]

The natural boundary conditions are

$$f_{y'}(a, y_0(a), y'_0(a)) = 0 \quad \text{and}$$

$$f_{y'}(b, y_0(b), y'_0(b)) = 0.$$

Case 2:

Suppose that both the end points are free to vary along continuous curves that possess continuous first derivatives. Let the two curves be $c_1(x)$ and $c_2(x)$. In this case the *transversality condition* must be satisfied.

Definition 1.5 [5, Page 332]

The transversality conditions are

$$f_{y'}[a, y_0(a), y'_0(a)][c'_1(a) - y'_0(a)] + f(a, y_0(a), y'_0(a)) = 0 \quad \text{and}$$

$$f_{y'}[b, y_0(b), y'_0(b)][c'_1(b) - y'_0(b)] + f(b, y_0(b), y'_0(b)) = 0$$

where $(a, c_1(a))$ is the point at which the function leaves the initial curve $c_1(x)$ and $(b, c_2(b))$ is the point at which the function crosses (transverses) the curve $c_2(x)$.

Fixed Point-Variable and Variable Point –Fixed Point [5, Page 332]

Here we will discuss only the simplest case of one unknown function with one independent variable. For example, if the initial point is fixed at $(a, y(a))$ while the terminal point is free to vary along the curve $c_2(x)$ which is continuous and has a continuous first derivative, the boundary conditions are

$$y_0(a) = y(a) \quad \text{and}$$

$$f_{y'}[b, y_0(b), y'_0(b)][c'_1(b) - y'_0(b)] + f(b, y_0(b), y'_0(b)) = 0.$$

Along with boundary conditions that are set on the functional in variational problems, the model range is also limited with *constraints*.

1.6 CONSTRAINTS

A constraint is a restriction or limitation on the set over which the functional may be minimized or maximized. This constraint may range from a simple identity to a model defined by a system of equations. In solving any application there will always be restrictions, and in order to get a meaningful result the problem will need to be solved within these restrictions.

In calculus of variations the method of *Lagrange multipliers* is a powerful tool for solving problems of constrained maxima and minima. In this method the Lagrangian function is formed and the unconstrained problem is solved using the appropriate forms of Euler equation.

Constrained problems are classified by the nature of the constraint. There are three types of problems, which are classified on the basis of the constraint by which they are governed. They are *Lagrange, Isoperimetric and Mayer problems*.

Lagrange Problem [5, Page 338]

The Lagrange problem can be stated as follows:

$$\text{Max(min): } F[y] = \int_a^b f(x, y, y') dx$$

subject to

$$\phi(x, y, y') = 0.$$

Definition 1.6[5, Page 338]

When the constraints are differential equations, the problem is a Lagrange problem.

To solve the problem, we define a new function $G(x, y, y') = f + \lambda(x)\phi$, where $\lambda(x)$ is an unknown function of x , *the Lagrangian multiplier*. The problem is reduced to finding

the function y and λ that minimizes or maximizes $\int_a^b G(x, y, y')dx$. The Euler-Lagrange

equation becomes

$$G_y - \frac{d}{dx}G_{y'} = 0.$$

Isoperimetric Problem

The isoperimetric problem is stated as follows:

$$\text{Max(min)} F[Y] = \int_a^b f(x, y, y')dx$$

subject to

$$J = \int_a^b G(x, y, y')dx$$

where J is a known constant.

Definition 1.7 [5, Page 338]

If the constraints are integrals, the problem is generally known as an isoperimetric problem.

To solve this problem the Lagrangian function $L(x, y, y')$ is formed as

$$L(x, y, y', \lambda) = F(x, y, y') + \lambda G(x, y, y')$$

where λ , the **Lagrange multiplier**, is a constant and each constraint has a Lagrange multiplier when forming the Lagrangian function. The Euler-Lagrange equation for one integral constraint is

$$\frac{d}{dx} \left(\frac{dL}{dy'} \right) - \frac{dL}{dy} = 0.$$

Mayer Problem [5, Page 338]

A Mayer problem is one in which there are n unknown functions and m constraints that are either differential or ordinary equations. For a Mayer problem, m is less than n .

To connect such a problem to the general Lagrange problems, define $n-m$ new functions $z_i(x)$ by $z_i = y'_i$. The problem is then

$$\max (\min): F[z] = \int_a^b (z_1 + z_2 + \dots + z_{n-m}) dx$$

subject to original constraints $\phi_1, \phi_2, \dots, \phi_m = 0$. The general technique is the same that we use with solving Lagrange problems.

In my thesis we will be dealing with Lagrange problems, as the constraint is differential equations in nature.

CHAPTER 2

APPLICATION OF CALCULUS OF VARIATION IN ECONOMICS

2.1 INTRODUCTION

Economists use mathematics in the exploration and exposition of economic ideas. Mathematics quantifies the relationship between economic variables among economic factors, which helps the economists to identify and analyze properties of economic systems. For this reason mathematics has become the language of modern analytical economics. The two branches of economics are Macro and Micro. Mathematical techniques are often used in the macroeconomics field as it deals with maximizing of the utility function in which the variables are constrained by equalities and inequalities. Thus constrained optimization problems are pivotal to economic theory, and Calculus of Variations are one of the mathematical tools that are used to solve the optimization models.

2.2 APPLICATION [4, Page 375]

The following case illustrates the use of calculus of variations in macroeconomics. In this example we will optimize the social welfare function. The restrictions on the objective functions are the relationships from the macroeconomic model. The aim is to derive the optimal time paths for gross national product (GNP), consumption, investment and government expenditure. This classic optimization problem is one in which calculus of variations can be used to solve this complex model. In this problem we use a version of the Phillips multiplier – accelerator model.

Using the notation from Allen [1960], the model is

$$Z = C + I + G \quad (1)$$

$$Y = \left[\frac{b}{D+b} \right] Z \quad (2)$$

$$C = (1-s) Y \quad (3)$$

$$I = vY' + A \quad (4)$$

where C= consumption

I = total investment

G = government expenditure

A = autonomous investment

Z = aggregate demand

Y = aggregate supply (GNP)

s = the saving ratio (S/Y)

b = the speed of response of supply to demand

v = the accelerator

D = the linear differential operator (d/dt).

Substituting equations (3) and (4) into (1)

$$Z = (1-s) Y + vY' + A + G$$

$$Y = \left[\frac{b}{D+b} \right] [(1-s) Y + vY' + A + G]$$

$$= \frac{bY - bsY + bY' + bA + bG}{D+b}$$

$$Y' + Yb - bY + bsY - bvY' - bA - bG = 0$$

$$Y' + bsY - bvY' - bA - bG = 0$$

$$Y' + bsY - bvY' - bA - bG = 0$$

$$(1-bv) Y' + bsY - bA - bG = 0.$$

The goal is to minimize

$$F = \int_t^T [a_1(Y - Y^*)^2 + a_2(G - G^*)^2] dt$$

subject to

$$(1-bv) Y' + bsY - bA - bG = 0$$

where Y^* = desired level of GNP,

G^* = desired level of government expenditure.

Constrained problems are classified by the nature of the constraint. In this model the constraint is a differential equation; therefore the problem is a **Lagrange problem**.

Another way of stating the above problem is

Minimize

$$F^* = \int_t^T [a_1(Y - Y^*)^2 + a_2(G - G^*)^2] dt + u \{ (1-bv) Y' + bsY - bG - bA \}$$

where u is called the **Lagrange multiplier**.

Now we know that the **Euler equation** [2, Page 365] is

$$\frac{dF}{dx} - \frac{d}{dt} \left[\frac{dF}{dx'} \right] = 0.$$

Using the Euler equation on F^* , we have

$$F^* = F + uV$$

$$F = a_1 (Y - Y^*)^2 + a_2 (G - G^*)^2$$

$$V = (1-bv) Y' + bsY - bG - bA = 0.$$

Now the ***Euler Lagrange equation*** is

$$\frac{dF^*}{dY} - \frac{d}{dt} \left[\frac{dF^*}{dY'} \right] = 0.$$

We calculate each term of the equation, we obtain the following:

$$\frac{dF^*}{dY} = \frac{dF}{dY} + u \frac{dV}{dY} + V \frac{du}{dY}$$

$$= 2a_1(Y - Y^*) + bsu$$

$$\frac{dF^*}{dY'} = \frac{dF}{dY'} + u \frac{dV}{dY'} + V \frac{du}{dY'}$$

$$= 0 + u(1 - bv)$$

$$= u(1 - bv)$$

$$\frac{d}{dt} \left[\frac{dF^*}{dY'} \right] = \frac{du}{dt}(1 - bv)$$

$$\frac{du}{dt}(1 - bv) = 2a_1(Y - Y^*) + bsu$$

$$\frac{du}{dt} = \frac{(2a_1(Y - Y^*) + bsu)}{(1 - bv)}. \quad (5)$$

The ***Euler equations*** for G and u are similarly determined in the following way.

$$\frac{dF^*}{dG} = \frac{dF}{dG} + u \frac{dv}{dG} + V \frac{du}{dG}$$

$$= 2a_2(G - G^*) + u(-b)$$

$$= 2a_2(G - G^*) - ub$$

$$\frac{dF^*}{dG'} = \frac{dF}{dG'} + u \frac{dv}{dG'} + V \frac{du}{dG'}.$$

$$= 0$$

Since F^* doesn't depend on G' and U' , we have

$$\frac{d}{dt} \left[\frac{dF^*}{dG'} \right] = 0$$

$$2a_2 (G - G^*) - bu = 0 \quad (6)$$

$$\frac{dF^*}{du} = \frac{dF}{du} + u \left[\frac{dv}{du} \right] + V$$

$$= 0 + 0 + (1 - bv)Y' + bsY - bG - bA$$

$$= (1 - bv) Y' + bsY - bG - bA$$

$$\frac{d}{dt} \left[\frac{dF^*}{du'} \right] = 0$$

$$(1 - bv) Y' + bsY - bG - bA = 0 . \quad (7)$$

Equations (5), (6) and (7) form a system of three equations in three unknowns

(Y , G and u) and their time derivatives. Solving equation (6), we have

$$2a_2 G - 2a_2 G^* = bu$$

$$2a_2 G = bu + 2a_2 G^*$$

$$G = \left(\frac{b}{2a_2} \right) u + G^* \quad (8)$$

and then substituting for G in (7), we obtain

$$(1 - bv) Y' + bsY - b \left\{ \left(\frac{b}{2a_2} \right) u + G^* \right\} - bA = 0$$

$$(1 - bv) Y' + bsY - \left(\frac{b^2}{2a_2} \right) u - bG^* - bA = 0. \quad (9)$$

Solving (5) for Y gives

$$Y = Y^* + (1-bv) \left(\frac{u'}{2a_1} \right) - \left(\frac{bs}{2a_1} \right) u. \quad (10)$$

Differentiating (10) with respect to t provides an equivalent for Y'

$$Y' = \left\{ (1-bv) \frac{u''}{2a_1} \right\} - \left(\frac{bsu'}{2a_1} \right). \quad (11)$$

Substituting (10) and (11) into (9) for Y and α respectively

$$\begin{aligned} & (1-bv) \left\{ (1-bv) \frac{u''}{2a_1} - \left(\frac{bsu'}{2a_1} \right) \right\} + bs \left\{ Y^* + (1-bv) \left(\frac{u'}{2a_1} \right) - \left(\frac{bsu'}{2a_1} \right) u \right\} \\ & \quad - \left(\frac{b^2u}{2a_2} \right) - bG^* - bA = 0 \\ & \left\{ \frac{(1-bv)^2}{2a_1} \right\} (u'') - \left\{ (1-bv) \left(\frac{bsu'}{2a_1} \right) \right\} + bsY^* + \left\{ (1-bv) \left(\frac{bsu'}{2a_1} \right) \right\} - \left(\frac{b^2s^2}{2a_1} \right) u \\ & \quad - \frac{b^2u}{2a_2} - bG^* - bA = 0 \\ & \left\{ \frac{(1-bv)^2}{2a_1} \right\} (u'') - b^2 \left\{ \left(\frac{s^2}{2a_1} \right) + \left(\frac{1}{2a_2} \right) \right\} u = -bsY^* + bG^* + bA \\ & \left\{ \frac{(1-bv)^2}{2a_1} \right\} (u'') - b^2 \left(s^2 + \frac{a_1}{a_2} \right) u = -2a_1b (sY^* - G^* - A). \end{aligned} \quad (12)$$

A **linear differential equation** is of the form [4, Page 366]

$$\sum_{i=1}^{k+1} a_i \frac{d^i x}{dt^i} + a_0 x + b_i + b_2 e^{rt} = 0$$

where $r, b_1, a_0, a_1, \dots, a_{k+1}$ are constants depending on the parameters of the problem.

In this equation x is a vector of m components, which would represent m differential

equations. The solution of each differential equation mentioned above is of the form

$$x = \sum_{i=1}^{k+1} A_i e^{\lambda_i t} + B_1 + B_2 e^{\rho t}$$

where

A_i = constants of integration,

B_1, B_2 and ρ = constants determined by a 's, b 's and r ,

λ = characteristic roots.

Here in equation (12) we will only need to calculate $B_1 = -\frac{b_1}{a_0}$. Now:

$$a_1 = \left\{ \frac{(1-bv)^2}{2a_1} \right\} D(\gamma)$$

$$a_0 = b^2 \left(s^2 + \frac{a_1}{a_2} \right) u$$

$$b_1 = \frac{2a_1(sY^* - G^* - A)}{b \left(s^2 + \frac{a_1}{a_2} \right)}.$$

The solution to equation (12) is

$$u = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + B_1. \quad (13)$$

The characteristic roots will be determined by solving

$$(1-bv)^2 \lambda^2 - b^2 \left\{ s^2 + \left(\frac{a_1}{a_2} \right) \right\} = 0$$

where the values of λ_1 and λ_2 for the two solutions are

$$\lambda_1 = \frac{b}{(1-bv)} \left(\sqrt{s^2 + \frac{a_1}{a_2}} \right)$$

$$\lambda_2 = \left(-\frac{b}{(1-bv)} \right) \left(\sqrt{s^2 + \frac{a_1}{a_2}} \right).$$

The optimal path for Y and G can be determined differentiating (13) with respect to t

$$\gamma = A_1 \lambda_1 e^{\lambda_1 t} + A_2 \lambda_2 e^{\lambda_2 t}. \quad (14)$$

The optimal path for Y is determined by substituting (13) and (14) into (10)

$$\begin{aligned} Y &= Y^* + \frac{(1-bv)}{2a_1} [A_1 \lambda_1 e^{\lambda_1 t} + A_2 \lambda_2 e^{\lambda_2 t}] - \frac{bs}{2a_1} [A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + B_1] \\ Y &= [Y^* + A_1 e^{\lambda_1 t} \left[\frac{(1-bv)}{2a_1} \lambda_1 - \frac{bs}{2a_1} \right] + A_2 e^{\lambda_2 t} \left[\frac{(1-bv)}{2a_1} \lambda_2 - \frac{bs}{2a_1} \right] \\ &\quad - \frac{bs}{2a_1} \left\{ 2a_1 \frac{(sY^* - G^* - A)}{b} \left(s^2 + \left(\frac{a_1}{a_2} \right) \right) \right\}] \\ \Omega &= [Y^* + A_1 e^{\lambda_1 t} \left\{ \frac{(1-bv)\lambda_1 - bs}{2a_1} \right\} + A_2 e^{\lambda_2 t} \left\{ \frac{(1-bv)\lambda_2 - bs}{2a_1} \right\} \\ &\quad + s \left\{ \frac{A + G - sY^*}{(s^2 + \frac{a_1}{a_2})} \right\}] \quad (15) \end{aligned}$$

The optimal path of G is found by putting equation (13) into (8)

$$\begin{aligned} G &= \frac{b}{2a_2} [A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + B_1] + G^* \\ &= \left(\frac{bA_1}{2a_2} \right) e^{\lambda_1 t} + \left(\frac{bA_2}{2a_2} \right) e^{\lambda_2 t} + \frac{2a_1 b}{2a_2 b} \left[\frac{(sY^* - G^* - A)}{(s^2 + (\frac{a_1}{a_2}))} \right] + G^* \\ &= \left(\frac{bA_1}{2a_2} \right) e^{\lambda_1 t} + \left(\frac{bA_2}{2a_2} \right) e^{\lambda_2 t} + \left(\frac{a_1}{a_2} \right) \left\{ \frac{(sY^* - G^* - A)}{(s^2 + (\frac{a_1}{a_2}))} \right\} + G^*. \end{aligned}$$

The optimal path for consumption and investment are found by substituting (15) into (3) and (4) respectively. Hence we have

$$\delta = (1-s) \Omega$$

$$\phi = v \frac{d\Omega}{dt} + A$$

where

$$\frac{d\Omega}{dt} = \lambda_1 \left(\frac{\lambda_1(1-bv)-bs}{2a_1} \right) A_1 e^{\lambda_1 t} + \lambda_2 \left(\frac{\lambda_2(1-bv)-bs}{2a_1} \right) A_2 e^{\lambda_2 t}.$$

2.3 FUTURE AIM

The validity of this model has been checked by taking into account data from the years 1900 to 1970. My future goal for this model is to numerically solve this model by taking the data from the years 1970-2004.

CHAPTER 3

ANALYSIS OF LUCAS MODEL

3.1 INTRODUCTION

Well-being of any country depends on the amount of physical capital, skills and knowledge of its citizens. The more capital we have the more output we can produce per worker. It is not hard to see why the labor force skills also matter. Highly skilled workers are more productive than less skilled workers. Consequently, an economy with a more skilled labor force is likely to grow faster than an economy with a less skilled labor force. Thus, economic growth will be determined by both physical and human capital accumulation.

Economists have created a number of models that describe how education might affect the growth rate of any economy. One such pioneering contribution was made by Lucas. (1988)

3.2 A BRIEF DESCRIPTION OF THE LUCAS MODEL [3, Page 3-42]

The focus of Lucas is how accumulation of human capital impacts growth rate of an economy. In his model, Lucas described an individual who allocates time either to production (i.e., work) or skill acquisition (education). Skill acquisition increases productivity in future periods. Hence, the more individuals study, the higher will be their productivity which would lead to faster economic growth. However, skill acquisition

comes at a penalty – the more time an individual spends on education, the less time is left for production.

Lucas assumes that the growth rate of the human capital is linearly related to its level. What if we abandon this assumption? Will the optimal value of the time devoted to education in the steady state exist? If it exists, will it be same or different?

Mathematically, relationship between human and physical capital along with the time allocated by an individual for production can be described with the following two equations:

$$y = Ak_t^\beta (u_t h_t)^\alpha,$$

$$\frac{dh}{dt} = ah_t(1 - u_t),$$

where

k_t = Physical capital stock

h_t = Human capital stock

u_t = Fraction of time that the individual allocates to production

α, β, A, a are positive constants.

The production function consisted of two components: human and physical capital. Since education and technology have a great impact on the production in any economy, economists have included these factors to the production function. Lucas in his production function also takes into account both the technology and education affect. He uses parameters A to describe overall technology level. The higher A , the more

productive is the individual. Parameters α and β determine marginal products of capital and human capital, respectively. Marginal product describes how much total output will increase if there is an increase of one unit of input.

We have taken u to be the time the individual spends on production. The implication is that $1-u$ is the time that would be devoted to education. The aim of the Lucas model was to find the optimal value $1-u$ in the steady state, i.e., when the rate of growth of all the variables is constant.

3.3 MODEL CONSTRUCTION

The aim of the new model is to investigate whether an optimal value of $1-u$ exists in a steady state if the rate of change of human capital is a nonlinear function.

The only modification we made to the Lucas model was in the equation that describes the process of human accumulation by introducing a nonlinear component. This nonlinear component captures the following idea- at initial stages of growth when the human capital is low, it might take a long time to learn something new. As H (slowly) grows, we reach a point of “enlightenment”. There is a jump in the growth rate of human capital. This new feature can be modeled by modifying the equation (2) in the following way:

$$\frac{dh}{dt} = (1 - u_t) \left(ah_t + \frac{b}{1 + e^{\gamma - \delta h_t}} \right) \quad (3)$$

where

b = Size of the jump,

γ = Location of the jump,

δ = Shape or abruptness of the jump.

In the above equation we have introduced three parameters related to jump. By jump we mean the point from where the change in accumulation of human capital related to education will be much higher than before that point; that is: spending a little more time on education gives a large increase in the human capital when compared to increase before that point.

Now if we substitute $b = 0$ or $\delta = 0$ (or both), we get the same linear function as did Lucas. But if b and δ , are non-zero then we get a jump in growth rate of h .

Change in the physical capital will be what the economy produces minus what it consumes. This is so because what is left is used in future production. So the constraint that the real economy faces in terms of physical capital is the following equation.

$$\frac{dk}{dt} = F(k, h) - c_t \quad (4)$$

Substituting equation (1) in (4), we get

$$\frac{dk}{dt} = A k_t^\beta (u_t h_t)^\alpha - c_t$$

Equation (5) describes how the rate of change of capital stock depends on production function, consumption and depreciation rate of capital, and this equation is the physical capital constraint. Lucas includes depreciation rate of the capital into the growth rate of the physical capital. He assumes that machinery depreciates due to continuous use and its productivity goes down. The depreciation is therefore described as a negative rate of change of physical capital stock. Thus the rate of change of physical capital is given by the following equation.

$$\frac{dk}{dt} = A k_t^\beta (u_t h_t)^\alpha - c_t - n k_t \quad (5)$$

where

n = Depreciation rate.

The model follows the same instantaneous utility function as that of the Lucas model and the function is given by the following equation.

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \text{ where } \sigma \text{ is the coefficient of relative risk aversion.}$$

Therefore, an individual is trying to maximize his lifetime utility subject to human and physical capital constraints:

$$\max F = \int_0^{\infty} \frac{c_t^{1-\sigma}}{1-\sigma} e^{\rho t} dt, \text{ where } \rho \text{ is the discount rate.}$$

subject to

$$k'_t - A k_t^{\alpha} (u_t h_t)^{\beta} + c_t + n k_t = 0$$

$$h'_t - (1 - u_t) \left(a h_t + \frac{b}{1 + e^{\gamma - \delta h_t}} \right) = 0.$$

To solve this economic model we will use calculus of variations. Mathematically, the problem is to maximize the functional which is subject to two differential equations constraints. Since we have four unknown functions and two constraints, this problem can be classified as a Mayer problem.

To solve this problem we will find the Euler-Lagrange equations for c , k , h and u thus we will get a system of equations. These equations will be solved to get c and u in terms of h and k . On solving for u our first aim would be to check if the value $u = 1$ and $\frac{du}{dt} = 0$ is possible. If so then we can say that the human and physical accumulation will remain constant, as h and k no longer depend on time, and if that it is possible then the optimal

behavior for an individual could be not to educate himself, which would contradict the Lucas model.

It is clear that c , h , k and u are all dependent on time; therefore, to simplify the notation we will drop the subscript t .

Thus the problem is

$$\text{maximize } F = \int_0^{\infty} \frac{c^{1-\sigma}}{1-\sigma} e^{\rho t} dt$$

subject to

$$k' - Ak^{\alpha}(uh)^{\beta} + c + nk = 0$$

$$h' - (1-u)(ah + \frac{b}{1+e^{\gamma-\delta h}}) = 0.$$

First we define

$$F^* = F + u_1 V_1 + u_2 V_2,$$

where

$$F = \frac{c^{1-\sigma}}{1-\sigma} e^{\rho t}$$

$$V_1 = k' - Ak^{\alpha}(uh)^{\beta} + c + nk$$

$$V_2 = h' - (1-u)(ah + \frac{b}{1+e^{\gamma-\delta h}}).$$

Now the *Euler-Lagrange equation* is

$$\frac{dF^*}{dc} - \frac{d}{dt} \left[\frac{dF^*}{dc'} \right] = 0.$$

We calculate each term of the equation. We obtain the following:

$$\begin{aligned}\frac{dF^*}{dc} &= \frac{dF}{dc} + u_1 \frac{dV_1}{dc} + V_1 \frac{du_1}{dc} + u_2 \frac{dV_2}{dc} + V_2 \frac{du_2}{dc} \\ &= c^{-\sigma} e^{\rho\alpha} + u_1.\end{aligned}$$

$$\frac{d}{dt} \left[\frac{dF^*}{dc'} \right] = 0$$

$$c^{-\sigma} e^{\rho\alpha} + u_1 = 0 \quad (1)$$

The Euler Equation for h, k and u are similarly determined in the following way.

Equation for h:

$$\begin{aligned}\frac{dF^*}{dh} &= 0 + u_1 (-Ak^\alpha u^\beta \beta h^{\beta-1}) - u_2 (1-u) \left(a + \frac{b \delta e^{\delta h + \gamma}}{(e^{\delta h} + e^\gamma)^2} \right) \\ &= -u_1 (Ak^\alpha u^\beta \beta h^{\beta-1}) - u_2 (1-u) \left(a + \frac{b \delta e^{\delta h + \gamma}}{(e^{\delta h} + e^\gamma)^2} \right) \\ \frac{dF^*}{dh'} &= u_2 \\ \frac{d}{dt} \left[\frac{dF^*}{dh'} \right] &= \frac{du_2}{dt} \\ \frac{du_2}{dt} &= -u_1 (Ak^\alpha u^\beta \beta h^{\beta-1}) - u_2 (1-u) \left(a + \frac{b \delta e^{\delta h + \gamma}}{(e^{\delta h} + e^\gamma)^2} \right) \quad (2)\end{aligned}$$

Equation for k:

$$\begin{aligned}\frac{dF^*}{dk} &= 0 + u_1 (-A \alpha k^{\alpha-1} u^\beta h^\beta + n) + u_2 (0) \\ \frac{dF^*}{dk'} &= u_1\end{aligned}$$

$$\frac{d}{dt} \left[\frac{dF^*}{dk'} \right] = \frac{du_1}{dt}$$

$$\frac{du_1}{dt} = u_1 (-A \alpha k^{\alpha-1} u^\beta h^\beta + n) \quad (3)$$

Equation for u:

$$\begin{aligned} \frac{dF^*}{du} &= 0 + u_1 (-A k^\alpha \beta u^{\beta-1} h^\beta) + u_2 [-(-1)] (ah + \frac{b}{1 + e^{\gamma - \delta h}}) \\ &= -u_1 (A k^\alpha \beta u^{\beta-1} h^\beta) + u_2 (ah + \frac{b}{1 + e^{\gamma - \delta h}}) \\ \frac{d}{dt} \left[\frac{dF^*}{du'} \right] &= 0 \\ -u_1 (A k^\alpha \beta u^{\beta-1} h^\beta) + u_2 (ah + \frac{b}{1 + e^{\gamma - \delta h}}) &= 0 \end{aligned} \quad (4)$$

Equations (1), (2), (3) and (4) form a system of four equations in three unknowns (c, u_1 and u_2).

Solving equation (3), we have

$$u_1 = e^{(-A \alpha k^{\alpha-1} u^\beta h^\beta + n)t + c_1} \quad (5)$$

Substituting equation (5) in (1), we obtain

$$c^{-\sigma} e^{\rho t} = e^{(-A \alpha k^{\alpha-1} u^\beta h^\beta + n)t + c_1}$$

$$\frac{e^{\rho t}}{c^\sigma} = e^{(-A \alpha k^{\alpha-1} u^\beta h^\beta + n)t + c_1}$$

$$\frac{e^{\rho t}}{e^{(-A \alpha k^{\alpha-1} u^\beta h^\beta + n)t + c_1}} = c^\sigma$$

$$\left[\frac{e^{\rho\alpha}}{e^{(n-A\alpha k^{\alpha-1}u^{\beta}h^{\beta})t+c_1}} \right]^{\frac{1}{\sigma}} = c \quad (6)$$

Solving (2) for u_2 , we obtain the following:

$$u_2' = e^{(\frac{-\alpha z}{k}+n)t} \left[\frac{-\beta z}{h} \right] - u_2(1-u)s$$

where

$$z = (Ak^{\alpha}u^{\beta}h^{\beta})$$

$$s = \left(a + \frac{b\delta e^{\delta\eta+\gamma}}{(e^{\delta\eta} + e^{\gamma})^2} \right)$$

$$u_2' + u_2(1-u)s = e^{(\frac{-\alpha z}{k}+n)t} \left[\frac{-\beta z}{h} \right]$$

$$\left(e^{(1-u)st} u_2 \right)' = e^{(1-us-\frac{\alpha z}{k}+n)t} \left[\frac{-\beta z}{h} \right]$$

$$u_2 = \frac{e^{(1-us-\frac{\alpha z}{k}+n)t} \left[\frac{-\beta z}{h} \right]}{(1-us-\frac{\alpha z}{k}+n)e^{(1-u)st}}$$

$$u_2 = \frac{e^{-ust} e^{(1-\frac{\alpha z}{k}+n)t} \left[\frac{-\beta z}{h} \right]}{(1-us-\frac{\alpha z}{k}+n)e^{st} e^{-ust}}$$

$$u_2 = \frac{e^{(1-\frac{\alpha z}{k}+n)t} \left[\frac{-\beta z}{h} \right]}{(1-us-\frac{\alpha z}{k}+n)e^{st}} \quad (7)$$

Substituting the equations (5) and (7) in (4), we have

$$e^{(\frac{-\alpha z}{k}+n)t} \left[\frac{-\beta z}{u} \right] + \frac{e^t e^{(\frac{-\alpha z}{k}+n)t} \left[\frac{-\beta z}{h} \right]}{(1-us-\frac{\alpha z}{k}+n)e^{st}} V = 0$$

where

$$V = \left(ah + \frac{b}{1 + e^{\gamma - \delta t}} \right).$$

The expressions $e^{(\frac{-\alpha z}{k}+n)t}$ and $[-\beta z]$ are common and therefore the above equation

reduces to

$$\frac{1}{u} + \frac{e^t}{h} \left[\frac{1}{(1-us-\frac{\alpha z}{k}+n)e^{st}} \right] V = 0$$

$$\frac{1}{u} (1-us-\frac{\alpha z}{k}+n)e^{st} + \frac{e^t V}{h} = 0$$

$$\left[\frac{1}{u} - \frac{\alpha z}{ku} + \frac{n}{u} \right] e^{st} = se^{st} - \frac{e^t V}{h}$$

$$\left[\frac{1}{u} - \frac{\alpha A k^\alpha u^\beta h^\beta}{ku} + \frac{n}{u} \right] e^{st} = se^{st} - \frac{e^t V}{h}$$

$$\frac{e^{st}}{u} - \frac{\alpha A k^{\alpha-1} u^\beta h^\beta e^{st}}{u} + \frac{n e^{st}}{u} = se^{st} - \frac{e^t V}{h}$$

$$e^{st} - \alpha A k^{\alpha-1} u^\beta h^\beta e^{st} + n e^{st} = (se^{st} - \frac{e^t V}{h}) u$$

$$(se^{st} - \frac{e^t V}{h}) u + \alpha A k^{\alpha-1} u^\beta h^\beta e^{st} - (n+1)e^{st} = 0.$$

Now differentiating u with respect to t, we have

$$(se^{st} - \frac{e^t V}{h}) \frac{du}{dt} + u(s^2 e^{st} - \frac{e^t V}{h}) + \alpha A k^{\alpha-1} h^\beta [u^\beta se^{st} + e^{st} \beta u^{\beta-1} \frac{du}{dt}] - (n+1)se^{st} = 0.$$

Dividing the equation by e^{st} , we get

$$\begin{aligned}
 (s - \frac{e^t V}{he^{st}}) \frac{du}{dt} + u(s^2 - \frac{e^t V}{he^{st}}) + \alpha A k^{\alpha-1} h^\beta [u^\beta s + \beta u^{\beta-1} \frac{du}{dt}] - (n+1)s &= 0 \\
 (se^{st} - \frac{eV}{he^{st}} + \beta u^{\beta-1}) \frac{du}{dt} &= (n+1)s - \alpha A k^{\alpha-1} h^\beta [u^\beta s] - u(s^2 - \frac{e^t V}{he^{st}}) \\
 \frac{du}{dt} &= \frac{(n+1)s - \alpha A k^{\alpha-1} h^\beta [u^\beta s] - u(s^2 - \frac{e^t V}{he^{st}})}{(se^{st} - \frac{eV}{he^{st}} + \beta u^{\beta-1})}. \tag{8}
 \end{aligned}$$

Considering the case where $u=1$ and $n=0$, and substituting in the equation,

$$\begin{aligned}
 (1 - s - \frac{\alpha A k^\alpha h^\beta}{k}) e^{st} + \frac{e^t V}{h} &= 0 \\
 (-1 + s + \frac{\alpha A k^\alpha h^\beta}{k}) &= \frac{e^t V}{he^{st}} \\
 (-1 + s + \frac{\alpha A k^\alpha h^\beta}{k}) &= \frac{e^{(1-s)t} V}{h}.
 \end{aligned}$$

As $u=1$,

$$\frac{du}{dt} = 0.$$

Therefore equation (8) reduces to

$$\begin{aligned}
 s - \alpha A k^{\alpha-1} h^\beta s - (s^2 - (s + \alpha A k^{\alpha-1} h^\beta - 1)) &= 0 \\
 s - \alpha A k^{\alpha-1} h^\beta s - s^2 + s + \alpha A k^{\alpha-1} h^\beta - 1 &= 0 \\
 s^2 - 2s + \alpha A k^{\alpha-1} h^\beta s - \alpha A k^{\alpha-1} h^\beta + 1 &= 0 \\
 (s-1)(s + (\alpha A k^{\alpha-1} h^\beta - 1)) &= 0 \\
 s = 1 - \alpha A k^{\alpha-1} h^\beta \text{ or} \\
 s &= 1
 \end{aligned}$$

We know that

$$s = a + \frac{b\delta e^{\delta h + \gamma}}{(e^{\delta h} + e^{\gamma})^2}$$

That means,

$$1 = a + \frac{b\delta e^{\delta h + \gamma}}{(e^{\delta h} + e^{\gamma})^2}$$

$$1 - a = \frac{b\delta e^{\delta h + \gamma}}{(e^{\delta h})^2 + (e^{\gamma})^2 + 2e^{\delta h}e^{\gamma}}$$

$$(1 - a)[(e^{\delta h})^2 + (e^{\gamma})^2 + 2e^{\delta h}e^{\gamma}] = b\delta e^{\delta h + \gamma}$$

$$(1 - a)(e^{\delta h})^2 + (1 - a)(e^{\gamma})^2 - (b\delta e^{\gamma} - (1 - a)2e^{\gamma})e^{\delta h} = 0$$

$$e^{\delta h} = \frac{(b\delta e^{\gamma} - (1 - a)2e^{\gamma}) \pm \sqrt{(b\delta e^{\gamma} - (1 - a)2e^{\gamma})^2 - 4(1 - a)(1 - a)2e^{\gamma}}}{2(1 - a)}$$

$$\delta h = \ln \left[\frac{(b\delta e^{\gamma} - (1 - a)2e^{\gamma}) \pm \sqrt{(b\delta e^{\gamma} - (1 - a)2e^{\gamma})^2 - 4(1 - a)(1 - a)2e^{\gamma}}}{2(1 - a)} \right]$$

$$h = \frac{1}{\delta} \ln \left[\frac{(b\delta e^{\gamma} - (1 - a)2e^{\gamma}) \pm \sqrt{(b\delta e^{\gamma} - (1 - a)2e^{\gamma})^2 - 4(1 - a)(1 - a)2e^{\gamma}}}{2(1 - a)} \right].$$

As h is not dependent on time it will remain constant.

Now taking the other root of s and solving for k

$$1 - \alpha A k^{\alpha-1} h^{\beta} = a + \frac{b\delta e^{\delta h + \gamma}}{(e^{\delta h} + e^{\gamma})^2}$$

$$\alpha A k^{\alpha-1} h^{\beta} = 1 - a - \frac{b\delta e^{\delta h + \gamma}}{(e^{\delta h} + e^{\gamma})^2}$$

$$k^{\alpha-1} = \frac{1}{\alpha A h^{\beta}} \left[1 - a - \frac{b\delta e^{\delta h + \gamma}}{(e^{\delta h} + e^{\gamma})^2} \right]$$

$$k = \left[\frac{1}{\alpha A h^\beta} \left[1 - \alpha - \frac{b \delta e^{\delta n + \gamma}}{(e^{\delta n} + e^\gamma)^2} \right] \right]^{\alpha-1}.$$

Since $0 < \alpha < 1$ then $\alpha - 1 < 0$, therefore,

$$k = \frac{1}{\left[\frac{1}{\alpha A h^\beta} \left[1 - \alpha - \frac{b \delta e^{\delta n + \gamma}}{(e^{\delta n} + e^\gamma)^2} \right] \right]^{\alpha-1}}$$

Again as k is not dependent on time therefore it will remain constant.

It can be concluded that it is possible for $u = 1$ and $\frac{du}{dt} = 0$. Therefore the human and physical accumulation will remain constant as h and k no longer depend on time.

Thus, given the above values of h and k at $u=1$, i.e., when an individual just allocates his time to current production, we have shown that it is possible that optimal behavior for an individual can be not to educate himself.

3.4 FUTURE SCOPE

The future aim of this model is to find the optimal division between the two activities, production and education, i.e., to find the optimal value of $1-u$ in the steady state. By steady state we mean that the rate of growth of consumption, the capital stock and the level of human capital is constant, and u , the fraction of time spent on work, is constant.

CHAPTER 4

MATHEMATICAL THEORY

4.1 Admissible Functions

The fundamentals of the calculus of variations can be stated as follows: Given a functional J and a well-defined set of functions A , determine which functions in A yield the minimum value or maximum value of J . The set A is called the set of *admissible functions*.

Definition

The set of functions satisfying the constraints of a given variational problem is called a the admissible function.

Example 4.1[2, Page 108]

The set of all admissible functions might be the set of all continuous functions on the interval $[a, b]$, or the set of all continuously differentiable functions on $[a, b]$ satisfying the boundary condition $f(a)=0$.

Example 4.2[2, Page109]

Let A be the set of all continuously differentiable functions on the interval $a \leq x \leq b$, which satisfy the boundary conditions $y(a)=y_0$ and $y(b)=y_1$. Let J be the arclength functional on A defined by

$$J(y) = \int_a^b \sqrt{1 + y'(x)^2} dx .$$

To each y in A the functional J associates a real number that is the arclength of the curve $y=y(x)$ between the fixed points $P(a,y_0)$ and $Q(b,y_1)$.

4.2 Fundamental Lemma of Calculus of Variation[2, Page 122]

If $f(x)$ is a continuous on $[a,b]$ and if

$$\int_a^b f(x)h(x)dx = 0 \quad (1)$$

for every twice differentiable function h with $h(a)=0$ and $h(b)=0$, then $f(x)=0$ for $x \in [a,b]$.

Proof: In order to obtain a contradiction assume for some x_0 in (a, b) that $f(x) > 0$ for all x in some interval (x_1, x_2) containing x_0 . For h choose

$$h(x) = \begin{cases} (x - x_1)^3 (x - x_2)^3, & x_1 \leq x \leq x_2 \\ 0 & \text{otherwise} \end{cases}$$

The cube of factors $(x-x_1)$ and (x_2-x) appears so that h is smoothed out at x_1 and x_2 and is therefore of class C^2 . Then

$$\int_a^b f(x)h(x)dx = \int_{x_1}^{x_2} f(x)(x - x_1)^3 (x - x_2)^3 dx > 0$$

since f is positive on (x_1, x_2) . This contradicts (1) and the lemma is proved.

4.3 The Derivation of the Euler-Lagrange Equation[2, Page 123]

The simplest problem in the calculus of variation is to find a local minimum for the functional

$$J(y) = \int_a^b L(x, y, y') dx$$

where $y \in C^2[a, b]$ and $y(a)=y_0$ and $y(b)=y_1$. L is a given function that is twice continuously differentiable on $[a, b] \times R^2$.

We seek a necessary condition. Let y be a local minimum and h a twice continuously differentiable function satisfying $h(a)=h(b)=0$. Then $y + \varepsilon h$ is an admissible function and

$$J(y + \varepsilon h) = \int_a^b L(x, y + \varepsilon h, y' + \varepsilon h') dx.$$

Therefore

$$\begin{aligned} \frac{d}{d\varepsilon} J(y + \varepsilon h) &= \int_a^b L(x, y + \varepsilon h, y' + \varepsilon h') dx \\ &= \int_a^b \{L_y(x, y + \varepsilon h, y' + \varepsilon h')h + L_{y'}(x, y + \varepsilon h, y' + \varepsilon h')h'\} dx \end{aligned}$$

where L_y denotes $\frac{dL}{dy}$ and $L_{y'}$ denotes $\frac{dL}{dy'}$.

$$J(y, h) = \frac{d}{d\varepsilon} J(y + \varepsilon h) \big|_{\varepsilon=0} = \int_a^b \{L_y(a, y, y')h + L_{y'}(x, y, y')h'\} dx.$$

So a necessary condition for y to be a local minimum is

$$\int_a^b \{L_y(a, y, y')h + L_{y'}(x, y, y')h'\} dx = 0$$

for all $h \in C^2[a, b]$ with $h(a)=h(b)=0$. Now we will apply integration by parts to the above condition so that h can be eliminated and a condition for y can be formulated.

$$\int_a^b (L_{y'}(x, y, y') - \frac{d}{dx} L_y(x, y, y')) h dx + L_{y'}(x, y, y') h \big|_{x=a}^{x=b} = 0.$$

Since h vanishes at a and b , the boundary condition term vanishes and the necessary condition becomes

$$\int_a^b (L_y(x, y, y') - \frac{d}{dx} L_{y'}(x, y, y')) h dx = 0 \quad (2)$$

for all $h \in C^2[a, b]$ with $h(a) = h(b) = 0$. In order to effectively eliminate h we will now use the *fundamental lemma of calculus of variation*.

Applying the above lemma to equation (2)

$$L_y(x, y, y') - \frac{d}{dx} L_{y'}(x, y, y') = 0.$$

Above is the Euler –Lagrange equation which represents a necessary condition for a local minimum.

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